Worksheet for October 29 and October 31

Problems marked with an asterisk are to be placed in your math diary.

(1.*) Let B denote the solid sphere of radius one centered at the origin in $\mathbb{R}^3$. Find the average value of the distances of the points $P = (x, y, z) \in B$ from the origin. I find the answer somewhat surprising, do you ? Can you make any sense out of your answer?

We have a change of variables formula for functions of three variables, that uses a transformation $G(u, v, w)$ from the uvw-coordinate system to the xyz-coordinate system. Here we think of x, y, z as functions of u, v, w, i.e., $x = x(u, v, w), y = y(u, v, w)$ and $z = z(u, v, w)$.Equivalently

$$G(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$$

The volume *stretching factor* is also given by the absolute value of a Jacobian, $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$.

Thus, $dV = |\frac{\partial(x,y,z)}{\partial(u,v,w)}|du\ dv\ dw$,

As before, if we assume:

(i) All first order partials of the coordinate functions exist and are continuous.

(ii) $G(u, v, w)$ is 1-1 on the interior of a given domain W_0 in uvw-space.

we get the following change of variables theorem for triple integrals.

Change of Variables Theorem. For $f(x, y, z)$ continuous on the bounded region $B \subseteq \mathbb{R}^3$, and $G(u, v, w)$ as above, $\int\int\int_B f(x, y, z)\ dV$ can be computed as:

$$\int\int\int_{B_0} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dV,$$

where $G(B_0) = B$. As in the case of the two variable change of variable formula, the formula above suggests that small units of volume dV in the x, y, z coordinate system correspond to small units of volume times $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$ in the u, v, w coordinate system. We have already seen an example of changing variables for triple integrals, namely, spherical coordinates.

(2.*). What are the three variable versions of translation and linear transformation? Compute the Jacobians in each case.

(3.*) Writing writing x, y, z in terms of spherical coordinates find the Jacobian of the transformation that takes x, y, z to spherical coordinates.

(4.*) Use a change of variables to calculate vol$(B) = \int\int\int_B\ dV$, for B the solid ellipsoid $0 \leq \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$.

(5.*) Let B be the solid parapellopiped spanned by the vectors $\vec{i}, 2\vec{i} + 3\vec{j}, 4\vec{i} + 5\vec{j} + 6\vec{k}$, translated 2 units in the x direciton, 2 units in the y direction and 2 units in the z direction. Find a transformation G so that $G(C) = B$, where C is the unit cube $[0, 1] \times [0, 1] \times [0, 1]$.

Note: You should be able to figure out problems 4 and 5 by mimicking what we did in the two variable case.

(6.*) Find the Jacobian of the cylindrical coordinate transformation.

(7.) Let E be the region bounded below by the $r\theta$ -plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same: (a) $dz\ dr\ d\theta$; (b) $dr\ dz\ d\theta$; (c) $d\theta\ dz\ dr$.